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Complete integrability of the Korteweg–de Vries equation under perturbation around its solution: Lie–Bäcklund symmetry approach

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Abstract. It is shown that when the Korteweg–de Vries equation is perturbed about a particular solution the resulting evolution equations to all order of perturbations admit infinitely many Lie–Bäcklund symmetries. The corresponding commuting constants of motion are derived and thereby complete integrability can be established.

1. Introduction

In recent years there has been progress in understanding the geometrical and group theoretical properties of many soliton equations (Ablowitz and Segur 1981, Anderson and Ibragimov 1979, Bullough and Caudrey 1980, Fuchssteiner and Fokas 1981, Lakshmanan 1978, 1979). It is noted that these exactly solvable systems possess many interesting features in common such as Bäcklund transformations, an infinite number of constants of motion which are in involution, N-soliton solutions (Ablowitz and Segur 1981, Anderson and Ibragimov 1979, Bullough and Caudrey 1980), an infinite number of Lie–Bäcklund (LB) symmetries (Fuchssteiner and Fokas 1981), etc. Recently it has been observed by Case and Roos (1981) that when a completely integrable Hamiltonian system in $(1+1)$ dimensions is perturbed about a particular solution the resulting equations are also completely integrable. As the existence of an infinite number of constants of motion of the soliton equations is intimately connected with the existence of an infinite number of LB symmetries (Fokas 1979, 1980, Kumei 1977, Strampp 1982), it is natural to search for the same for the perturbed Hamiltonian equations as well. In this paper, we consider the Korteweg–de Vries (KdV) equation as a specific example and show the existence of an infinite number of LB symmetries for each order of the perturbation equation. It is also pointed out that the corresponding constants of motion which are in involution can be obtained straightaway using the relation between the LB symmetries and conserved covariants so that the complete integrability is established.

2. LB symmetries and KdV equation under perturbation around its solution

We consider the KdV equation in the form

$$U_t = -\partial_x(U^2+2U_{xx}). \tag{1}$$

In order to study the properties of the perturbed system of (1), as a first step, we perturb the particular solution $u^{(0)}$ of (1) in such a way that

$$U = u^{(0)} + \Delta u \equiv u^{(0)} + \sum_{k=1}^{n} \varepsilon^k u^{(k)} \tag{2}$$

where $\varepsilon \in R$ is a small real parameter. The given solution $u^{(0)}$ in (2) is, for example, the soliton solution of (1). Substituting (2) into (1) and collecting the coefficients of ε^i, setting each of them individually equal to zero, the resulting equations are of the form

$$\Omega^{(i)} = u_t^{(i)} + 2u_{xxx}^{(i)} + 2 \sum_{0 \leqslant j \leqslant i} u^{(j)} u_x^{(i-j)} = 0, \qquad i = 0, 1, 2, \ldots, n. \tag{3}$$

We note at this stage that further analysis of (3) is facilitated† by assigning a weight n to $u^{(n)}$ and its derivatives and requiring that each term in the equations derived from it must have equal weight.

Considering now the following vector-valued functions:

$$u = (u^{(0)}, u^{(1)}, \ldots, u^{(n)}), \tag{4a}$$

$$u_k = (u_k^{(0)}, u_k^{(1)}, \ldots, u_k^{(n)}), \qquad k = 0, 1, 2, \ldots, \infty, \tag{4b}$$

$$u_0^{(i)} = u^{(i)}, \qquad u_k^{(i)} = \frac{\partial^k u^{(i)}}{\partial x^k}, \qquad u_{kt}^{(i)} = \frac{\partial u_k^{(i)}}{\partial t}, \qquad i = 0, 1, 2, \ldots, n. \tag{4c}$$

We define the ring of functions $G^i(u)$, and a corresponding vector-valued function $G(u)$, in such a way that

$$G(u) = (G^{(0)}(u), G^{(1)}(u), \ldots, G^{(n)}(u)) \tag{4d}$$

where

$$\{G^{(i)}(u) | G^{(i)} = G^{(i)}(u, u_1, u_2, \ldots)\}. \tag{4e}$$

These functions are assumed to have the usual smoothness properties, defined over appropriate space, and to vanish at $\pm\infty$. Then (3) is a particular case of the vector-valued equation

$$\Omega \equiv u_t + K(u) = 0, \tag{5a}$$

where

$$\Omega = (\Omega^{(0)}, \Omega^{(1)}, \ldots, \Omega^{(n)}), \qquad K(u) = (K^{(0)}(u), K^{(1)}(u), \ldots, K^{(n)}(u)). \tag{5b, c}$$

It is easy to see that the soliton solution $u^{(0)}$ satisfies the KdV equation

$$\Omega^{(0)} = u_t^{(0)} + \partial_x (u^{(0)2} + 2u_2^{(0)}) = 0. \tag{6}$$

For our further discussions we also need the total derivative operators defined as

$$D = D_x = \sum_{i=0}^{n} \sum_{k=0}^{\infty} u_{k+1}^{(i)} \frac{\partial}{\partial u_k^{(i)}}, \qquad D_t = \sum_{i=0}^{n} \sum_{k=0}^{\infty} u_{kt}^{(i)} \frac{\partial}{\partial u_k^{(i)}}. \tag{7a, b}$$

† We thank the referee for pointing this out to us.

Then the LB operator is defined as

$$X(\eta)=\sum_{i=0}^{n}(D_t\eta^{(i)})\frac{\partial}{\partial u_t^{(i)}}+\sum_{i=0}^{n}\sum_{k=0}^{\infty}(D^k\eta^{(i)})\frac{\partial}{\partial u_k^{(i)}} \tag{8}$$

where the vector-valued generalised LB symmetries

$$\eta(u)=(\eta^{(0)}(u),\eta^{(1)}(u),\ldots,\eta^{(n)}(u)). \tag{9}$$

We note that the symmetry $\eta^{(i)}$ is associated with the corresponding $u^{(i)}$ of (3). It is clear then that the LB operator $X(\eta)$ in (8) leaves the system (3) invariant (Strampp 1982) iff

$$X(\eta)\Omega^{(i)}|_{\Omega^{(i)}=0}\doteq 0,\qquad i=0,1,\ldots,n, \tag{10}$$

where $\doteq$ stands for the restriction to solution of (3). From (10) and (3), using (7) and (8) the explicit form of the invariant equation is given by

$$D_t\eta^{(i)}+2D^3\eta^{(i)}+2\sum_{0\leqslant j\leqslant i}(u_1^{(i-j)}\eta^{(j)}+u^{(j)}D\eta^{(i-j)})=0,\qquad i=0,1,\ldots,n. \tag{11}$$

To prove the existence of an infinite number of commuting vector-valued symmetries η_l, $l=1,2,\ldots,\infty$, it is generally enough to find one generalised vector-valued LB symmetry apart from the already known space and time translation LB symmetries. This is because with the above symmetries it is often possible to construct the so-called strong symmetry and hereditary symmetry, which recursively generates new symmetries from the known ones (Fuchssteiner and Fokas 1981, Fokas 1979, 1980).

We now introduce the following operator-valued quantities. Let the Fréchet derivative of the vector-valued function $K(u)$ (see (5)) be defined by

$$K'(u)[v]=\frac{\partial K}{\partial\varepsilon}(u+\varepsilon v)\bigg|_{\varepsilon=0}. \tag{12}$$

It is clear that the operator $K'(u)$ is an $(n+1)\times(n+1)$ matrix given by

$$K'(u)=(K_j^{(i)}),\qquad \begin{cases} i=0,1,\ldots,n,\\ j=0,1,\ldots,n,\end{cases} \tag{13}$$

where

$$K_j^{(i)}=\sum_{k=0}^{\infty}\frac{\partial K^{(i)}}{\partial u_k^{(j)}}D^k,\qquad i=0,1,\ldots,n. \tag{14}$$

We further define the Fréchet derivative of an $(n+1)\times(n+1)$ matrix operator-valued function $\Phi(u)$ by

$$\Phi'(u)[v]w=\frac{\partial\Phi}{\partial\varepsilon}(u+\varepsilon v)w\bigg|_{\varepsilon=0} \tag{15}$$

where the operator

$$\Phi'(u)=(\Phi_j^{i'}(u)),\qquad i,j=0,1,\ldots,n, \tag{16}$$

and Φ_j^i is the (i,j)th component of Φ. In the above definitions the vector-valued functions v and w are arbitrary functions of u, u_k, $k=1,2,\ldots,\infty$. Let the matrix operator-valued function Φ be a map of the vector-valued symmetries η in such a way that

$$\eta_{l+1}=\Phi\eta_l, \qquad l=1,2,3,\ldots,\infty. \tag{17}$$

Then the operator Φ is a strong symmetry iff it satisfies the condition

$$\Phi'[K]v-[K',\Phi]v=0. \tag{18}$$

By the definitions (17) and (18) we have an infinite sequence of vector-valued LB symmetries. Further, the commutativity of these symmetries is guaranteed by the concept of hereditary symmetry. If the operator Φ in (17) satisfies

$$\Phi^2(v'[w]-w'[v])+\Phi\{(\Phi w)'[v]\}-v'[\Phi w]+[\Phi(v),\Phi(w)]-\Phi[(\Phi v)'[w]-w'[\Phi v]]=0, \tag{19}$$

where v and w are functions of u, $u_1,\ldots$, then we say that Φ is hereditary. By the hereditary of Φ we will also have the hierarchy of exactly solvable matrix-valued equations (Fokas 1980):

$$u_t+\Phi^m u_x=0, \qquad m=0,1,2,\ldots,\infty. \tag{20}$$

Since the x-translation symmetry is the first among the matrix-valued local LB symmetries (η_l), and then (20) with the aid of (17) can also be written as

$$u_t+\eta_{m+1}=0, \qquad m=0,1,2,\ldots,\infty. \tag{21}$$

Having obtained infinitely many commuting LB symmetries of (5), it is of importance to investigate the associated conserved quantities. Let the matrix functional

$$\begin{aligned} I_l&=\int_{-\infty}^{\infty}\rho_l(x,u^{(0)},u^{(1)},u_1^{(0)},\ldots)\,\mathrm{d}x\\ &=(I_l^{(0)},I_l^{(1)},\ldots,I_l^{(n)}), \qquad l=1,2,3,\ldots,\infty, \end{aligned} \tag{22}$$

be the associated constants of motion for the system of equations (5). Then the corresponding vector-valued conserved covariants $\gamma_l=(\gamma_l^{(0)},\gamma_l^{(1)},\ldots,\gamma_l^{(n)})$ can be written as

$$\gamma_l=\operatorname{grad}I_l\Leftrightarrow\gamma_l^{(s)}=\delta I_l^{(i)}/\delta u^{(j)}, \qquad l=1,2,3,\ldots,\infty, \qquad i=0,1,2,\ldots,n, \tag{23a}$$

such that $s=i-j$, $0\leqslant s, j\leqslant i$. More explicitly we have

$$(\gamma_l^{(0)},\gamma_l^{(1)},\ldots,\gamma_l^{(i)})=(\delta I_l^{(i)}/\delta u^{(i)},\delta I_l^{(i)}/\delta u^{(i-1)},\ldots,\delta I_l^{(i)}/\delta u^{(0)}) \tag{23b}$$

or, stated simply, for the nth perturbed equation, the component form of the conserved covariant γ corresponding to $u^{(j)}$ is given by $\delta I^{(n)}/\delta u^{(j)}\equiv\gamma^{(n-j)}$.

Now the equations of motion (5) with reference to (3), that is, the KdV equation (1) perturbed around its solution, can always be rewritten in the Hamiltonian form (Case and Roos 1981)

$$\begin{aligned} &u_t+J\delta\mathscr{H}/\delta u\Leftrightarrow(u_t^{(s)})=-J(\delta\mathscr{H}^{(i)}/\delta u^{(j)})\equiv-K^{(s)},\\ &s=i-j, \qquad 0\leqslant s,j\leqslant i, \qquad i=0,1,2,\ldots,n, \end{aligned} \tag{24}$$

where the matrix-valued Hamiltonian functional $\mathscr{H}(u)=(\mathscr{H}^{(0)}(u),\mathscr{H}^{(1)}(u),\ldots,\mathscr{H}^{(n)}(u))$ is obtained straightaway by perturbing the KdV Hamiltonian

$$\mathscr{H}(U)=\int_{-\infty}^{\infty}(U^3/3-U_x^2)\,\mathrm{d}x \tag{25}$$

of (1) and J is an $(i+1)\times(i+1)$ $(i=0,1,2,\ldots,n)$ skew-symmetric matrix operator with the diagonal elements equal to the total differential operator D. Equation (24) may also be written more explicitly in component form or matrix form as

$$u_t^{(i)} = -D\frac{\delta\mathcal{H}^{(n)}}{\delta u^{(n-i)}} \qquad \text{or} \qquad u_t = -J\gamma; \qquad \gamma^{(i)} = \frac{\delta\mathcal{H}^{(n)}}{\delta u^{(n-i)}}. \tag{24a}$$

Further, from the fact that equations (5) or (24) admit strong and hereditary symmetry and so the infinite hierarchy of evolution equations (20), any of the constants of motion of (5) or (24) can be used as a Hamiltonian. Accordingly, by using the definitions (23) and (24), we can rewrite the hierarchy (20) in the Hamiltonian form

$$u_t + J\delta I_{m+1}/\delta u = u_t + J\gamma_{m+1} = 0, \qquad m = 0, 1, 2, \ldots, \infty. \tag{26}$$

Comparing (21) and (26) we obtain the relation

$$J\gamma_{m+1} = \eta_{m+1}, \qquad m = 0, 1, 2, \ldots, \infty, \tag{27}$$

relating the vector-valued conserved covariants and the vector-valued LB symmetries.

Finally, we obtain a recursive relation for the conserved covariants through the adjoint of the strong symmetry. By definition, the matrix-valued adjoint operator Φ^+ satisfies the condition

$$\langle f, \Phi g\rangle = \langle \Phi^+ f, g\rangle \tag{28}$$

for the given vector-valued covariants f and symmetries g with respect to the product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} fg \, \mathrm{d}x = \int_{-\infty}^{\infty} \left(\sum_{r=0}^{n} f^{(r)} g^{(n-r)} \right) \mathrm{d}x. \tag{29}$$

Since I_l is a conserved quantity of (5), it is also a conserved quantity of the whole hierarchy (20), so that

$$\mathrm{d}I_l/\mathrm{d}t = I_l'(u_t) = 0 \Leftrightarrow \langle \gamma_l, K\rangle = 0,$$

as well as

$$\langle \gamma_l, \Phi^m K\rangle = 0, \qquad m = 0, 1, 2, \ldots, \infty.$$

From (29) we also have the relation $\langle \gamma_l, \Phi^m K\rangle = \langle (\Phi^+)^m \gamma_l, K\rangle$, and therefore

$$\gamma_{l+m} = (\Phi^+)^m \gamma_l, \qquad m = 0, 1, 2, \ldots, \infty, \tag{30}$$

are also the conserved covariants of (5). Finally from the relations (27) and (30) we readily establish that

$$\Phi\eta_l = \Phi J\gamma_l = \eta_{l+1} = J\gamma_{l+1} = J\Phi^+\gamma_l \tag{31}$$

so that

$$J\Phi^+ = \Phi J \tag{32}$$

which connects the strong symmetry and its adjoint.

3. First-order perturbation

We first consider the first-order perturbation equation $(i=1)$ in (3) in the form

$$u_t^{(1)} + 2u_3^{(1)} + 2u^{(0)}u_1^{(1)} + 2u^{(1)}u_1^{(0)} = 0, \tag{33}$$

find its infinite number of commuting LB symmetries and obtain the corresponding constants of motion, and then show possible generalisation of these results to any order of perturbations.

In order to find the LB symmetries of (33) we have to consider this in conjunction with the LB symmetries of the corresponding unperturbed form (6) ($i=0$ in (3)). The various LB symmetries of (6) are obtained by solving its LB invariant equation ($i=0$ in (11))

$$D_t\eta^{(0)}+2D^3\eta^{(0)}+2u^{(0)}D\eta^{(0)}+2u_1^{(0)}\eta^{(0)}=0 \tag{34}$$

recursively. For example, we have the following first three LB symmetries of (6) (Fokas 1980):

$$\eta_1^{(0)}=u_1^{(0)}, \qquad \eta_2^{(0)}=2u_3^{(0)}+2u^{(0)}u_1^{(0)}, \tag{35a, b}$$

$$\eta_3^{(0)}=4u_5^{(0)}+\tfrac{40}{3}u_1^{(0)}u_2^{(0)}+\tfrac{20}{3}u^{(0)}u_3^{(0)}+\tfrac{10}{3}(u^{(0)})^2u_1^{(0)}. \tag{35c}$$

Using these three LB symmetries it is possible to construct the strong and hereditary symmetry $\Phi^{(0)}$ of (6) which satisfies the conditions (18) and (19). This operator $\Phi^{(0)}$ is given by (Fokas 1980)

$$\Phi=\Phi^{(0)}=2D^2+\tfrac{4}{3}u^{(0)}+\tfrac{2}{3}u_1^{(0)}D^{-1}, \tag{36a}$$

where

$$D^{-1}w(x)=\int_{-\infty}^{x}w(y)\,dy \tag{36b}$$

generates an infinite number of commuting LB symmetries $\{\eta_l^{(0)}\}$, $l=1,2,\ldots,\infty$ for (6).

The action of the LB operator (8) on the first-order perturbation equation (33) gives the corresponding invariant equation ($i=1$ in (11))

$$D_t\eta^{(1)}+2D^3\eta^{(1)}+2u^{(0)}D\eta^{(1)}+2u^{(1)}D\eta^{(0)}+2u_1^{(1)}\eta^{(0)}+2u_1^{(0)}\eta^{(1)}=0. \tag{37}$$

In order to find the exact form of the symmetries $\eta^{(1)}$, we assume that $\eta^{(1)}$ does not contain x and t explicitly. The first two symmetries corresponding to space and time translations of (33) are written readily as

$$\eta_1^{(1)}=u_1^{(1)}, \tag{38}$$

$$\eta_2^{(1)}=2u_3^{(1)}+2u^{(0)}u_1^{(1)}+2u_1^{(0)}u^{(1)}. \tag{39}$$

To find a more general symmetry we proceed as follows (Fokas 1980, Tamizhmani and Lakshmanan 1982). Calling $\eta_3^{(1)}=\xi$, we search for a

$$\xi=\xi[u^{(0)},u_1^{(0)},u_2^{(0)},u_3^{(0)},u_4^{(0)},u_5^{(0)},u^{(1)},u_1^{(1)},u_2^{(1)},u_3^{(1)},u_4^{(1)},u_5^{(1)}], \tag{40}$$

with unit weight. Substituting (40) in (37) and eliminating $u_{it}^{(0)}$ and $u_{it}^{(1)}$, $i=0,1,2,3,4,5$, by using (6) and (33), we collect the coefficients of $u_7^{(1)}$, $u_7^{(0)}$, $u_6^{(1)}$, $u_6^{(0)}$ (noting that the coefficients of $u_8^{(1)}$ and $u_8^{(0)}$ are cancelled). Then equating each of these coefficients to zero, we obtain

$$D\xi_{u_5^{(1)}}=0, \qquad D\xi_{u_4^{(1)}}=0. \tag{41}$$

Solving these equations we have

$$\xi=a_1u_5^{(1)}+a_2u_4^{(1)}+A(u^{(0)},u_1^{(0)},u_2^{(0)},u_3^{(0)},u^{(1)},u_2^{(1)},u_3^{(1)}) \tag{42}$$

where a_1 and a_2 are arbitrary integration constants. Substituting (42) in the remaining terms of the expanded version of (37) and equating the coefficients of $u_5^{(0)}$ and $u_5^{(1)}$ to zero, we have

$$DA_{u_3^{(0)}} = 2a_1u_1^{(1)} - \tfrac{4}{3}u_1^{(1)} + \tfrac{1}{3}a_2u^{(1)}, \tag{43a}$$

$$DA_{u_3^{(1)}} = \tfrac{5}{3}a_1u_1^{(0)}. \tag{43b}$$

Solving (43a) and (43b), we get $a_2 = 0$ and

$$A = 2a_1u^{(1)}u_3^{(0)} - \tfrac{4}{3}u^{(1)}u_3^{(0)} + \tfrac{5}{3}a_1u^{(0)}u_3^{(1)} + a_3u_3^{(1)} + B(u^{(0)}, u_1^{(0)}, u_2^{(0)}, u^{(1)}, u_1^{(1)}, u_2^{(1)}) \tag{44}$$

where a_3 is again an integration constant.

Repeating the same procedure and equating the coefficients of $u_4^{(0)}$ and $u_4^{(1)}$ to zero and solving the determining equations we find that $a_1 = 4$, $a_3 = 0$ and

$$B = \tfrac{40}{3}u_1^{(1)}u_2^{(0)} + \tfrac{40}{3}u_1^{(0)}u_2^{(1)} + a_4u_2^{(1)} + C(u^{(0)}, u_1^{(0)}, u^{(1)}, u_1^{(1)}) \tag{45}$$

where a_4 is an integration constant. Proceeding in this way, we find the final form of the LB symmetry as

$$\xi = 4u_5^{(1)} + \tfrac{20}{3}u_3^{(0)}u^{(1)} + \tfrac{20}{3}u^{(0)}u_3^{(1)} + \tfrac{40}{3}u_2^{(0)}u_1^{(1)} + \tfrac{40}{3}u_1^{(0)}u_2^{(1)} + \tfrac{20}{3}u^{(0)}u_1^{(0)}u^{(1)} + \tfrac{10}{3}(u^{(0)})^2u_1^{(1)} \tag{46}$$

with the constant $a_4 = 0$. This symmetry is a solution of (37) (note that in (37), $\eta^{(0)}$ is already known).

From the knowledge of the above first three LB symmetries of zeroth- and first-order equations, that is, $\eta_1^{(0)}$, $\eta_2^{(0)}$, $\eta_3^{(0)}$ and $\eta_1^{(1)}$, $\eta_2^{(1)}$, $\eta_3^{(1)}$, we can find an operator

$$\Phi^{(1)} = \begin{bmatrix} \Phi^{(0)}(u^{(0)}) & 0 \\ \Phi^{(0)\prime}(u^{(1)}) & \Phi^{(0)}(u^{(0)}) \end{bmatrix} \equiv \begin{bmatrix} 2D^2 + \tfrac{4}{3}u^{(0)} + \tfrac{2}{3}u_1^{(0)}D^{-1} & 0 \\ \tfrac{4}{3}u^{(1)} + \tfrac{2}{3}u_1^{(1)}D^{-1} & 2D^2 + \tfrac{4}{3}u^{(0)} + \tfrac{2}{3}u_1^{(0)}D^{-1} \end{bmatrix} \tag{47}$$

which satisfies the strong symmetry condition (18):

$$\Phi^{(1)\prime}[K]\psi - [K', \Phi^{(1)}]\psi = 0, \qquad K = (K^{(0)}, K^{(1)}), \qquad \psi = (\psi_1, \psi_2), \tag{48a}$$

$$K^{(0)} = +2u_3^{(0)} + 2u^{(0)}u_1^{(0)}, \qquad K^{(1)} = +2u_3^{(1)} + 2u^{(0)}u_1^{(1)} + 2u^{(1)}u_1^{(0)}. \tag{48b}$$

In fact, we verify that

$$\Phi^{(1)\prime}[K]\psi = \begin{bmatrix} \tfrac{4}{3}K^{(0)} + \tfrac{2}{3}DK^{(0)}D^{-1} & 0 \\ \tfrac{4}{3}K^{(1)} + \tfrac{2}{3}DK^{(1)}D^{-1} & \tfrac{4}{3}K^{(0)} + \tfrac{2}{3}DK^{(0)}D^{-1} \end{bmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$K'[\Phi^{(1)}\psi] = \begin{bmatrix} 2D^3 + 2u_1^{(0)} + 2u^{(0)}D & 0 \\ 2u_1^{(1)} + 2u^{(1)}D & 2D^3 + 2u^{(0)}D + 2u_1^{(0)} \end{bmatrix}$$

$$\times \begin{pmatrix} 2\psi_{1xx} + \tfrac{4}{3}u^{(0)}\psi_1 + \tfrac{2}{3}u_1^{(0)}D^{-1}\psi_1 \\ \tfrac{4}{3}u^{(1)}\psi_1 + \tfrac{2}{3}u_1^{(1)}D^{-1}\psi_1 + 2\psi_{2xx} + \tfrac{4}{3}u^{(0)}\psi_2 + \tfrac{2}{3}u_1^{(0)}D^{-1}\psi_2 \end{pmatrix}, \tag{50}$$

$$\Phi^{(1)}K'[\psi) = \begin{bmatrix} 2D^2 + \tfrac{4}{3}u^{(0)} + \tfrac{2}{3}u_1^{(0)}D^{-1} & 0 \\ \tfrac{4}{3}u^{(1)} + \tfrac{2}{3}u_1^{(1)}D^{-1} & 2D^2 + \tfrac{4}{3}u^{(0)} + \tfrac{2}{3}u_1^{(0)}D^{-1} \end{bmatrix}$$

$$\times \begin{pmatrix} 2\psi_{1xxx} + 2u_1^{(0)}\psi_1 + 2u^{(0)}\psi_{1x} \\ 2u_1^{(1)}\psi_1 + 2u^{(1)}\psi_{1x} + 2\psi_{2xxx} + 2u^{(0)}\psi_{2x} + 2u_1^{(0)}\psi_2 \end{pmatrix} \tag{51}$$

so that (48) holds good. This strong symmetry operator $\Phi^{(1)}$ therefore generates an infinite number of LB symmetries of both (6) and (33) in such a way that

$$\begin{pmatrix}\eta_{l+1}^{(0)}\\ \eta_{l+1}^{(1)}\end{pmatrix}=\Phi^{(1)}\begin{pmatrix}\eta_{l}^{(0)}\\ \eta_{l}^{(1)}\end{pmatrix},\qquad l=1,2,3,\ldots,\infty,\tag{52}$$

where $\Phi^{(1)}$ is given by (47) and $\eta_l^{(0)}$ and $\eta_l^{(1)}$ are LB symmetries of (6) and (33) respectively.

Similar to the above verification of the strong symmetry condition we can evaluate each term of (19) by taking the Fréchet derivatives of $\Phi^{(1)}$ along the direction of v, w and Φv and Φw, and prove that (19) is satisfied and hence $\Phi^{(1)}$ is a hereditary symmetry as well for the first-order perturbation equation (33) in conjunction with (6). As discussed in § 2 this leads to the associated hierarchy of evolution equations (20) and (21) and the consequences thereof which we discuss below.

4. Constants of motion for the first-order perturbation equation

From the infinitely many LB symmetries obtained in § 3, we are able to generate the corresponding constants of motion. For this purpose we note from (32) that the strong symmetry $\Phi^{(1)}$ satisfies the relation

$$J\Phi^{(1)+}=\Phi^{(1)}J\tag{53}$$

where

$$J=\begin{pmatrix}D & 0\\ 0 & D\end{pmatrix}\tag{54}$$

is the 2×2 skew-symmetric diagonal matrix operator with the diagonal elements equal to the total derivative operator D and $\Phi^{(1)+}$ is the adjoint of $\Phi^{(1)}$ as defined in (29). Since J is invertible, the adjoint operator can be written in the form

$$\Phi^{(1)+}=J^{-1}\Phi^{(1)}J.\tag{55}$$

From (47) and (55) we find that

$$\Phi^{(1)+}=\begin{bmatrix}2D^2+\frac{4}{3}D^{-1}u^{(0)}D+\frac{2}{3}D^{-1}u_1^{(0)} & 0\\ \frac{4}{3}D^{-1}u^{(1)}D+\frac{2}{3}D^{-1}u_1^{(1)} & 2D^2+\frac{4}{3}D^{-1}u^{(0)}D+\frac{2}{3}D^{-1}u_1^{(0)}\end{bmatrix}.\tag{56}$$

Then the infinitely many conserved covariants can be obtained from the relation (30), which now reads

$$\begin{pmatrix}\gamma_{l+1}^{(0)}\\ \gamma_{l+1}^{(1)}\end{pmatrix}=\Phi^{(1)+}\begin{pmatrix}\gamma_{l}^{(0)}\\ \gamma_{l}^{(1)}\end{pmatrix},\qquad l=1,2,3,\ldots,\infty.\tag{57}$$

Explicitly, we find the conserved covariants of (6) and (33) as

$$\gamma_1^{(0)}=u^{(0)},\qquad \gamma_2^{(0)}=2u_2^{(0)}+(u^{(0)})^2,\tag{58a, b}$$

$$\gamma_3^{(0)}=4u_4^{(0)}+\tfrac{20}{3}u^{(0)}u_2^{(0)}+\tfrac{10}{3}(u_1^{(0)})^2+\tfrac{10}{9}(u^{(0)})^3,\qquad \text{etc,}\tag{58c}$$

$$\gamma_1^{(1)}=u^{(1)},\qquad \gamma_2^{(1)}=2u_2^{(1)}+2u^{(0)}u^{(1)},\tag{59a, b}$$

$$\gamma_3^{(1)}=4u_4^{(1)}+\tfrac{20}{3}u^{(1)}u_2^{(0)}+\tfrac{20}{3}u^{(0)}u_2^{(1)}-\tfrac{20}{3}u_1^{(0)}u_1^{(1)}+\tfrac{10}{3}(u^{(0)})^2u^{(1)},\qquad \text{etc.}\tag{59c}$$

Further, using (23) (for $0\leq s, j\leq 1$) in the case of the first-order perturbation equation (in conjunction with (6)), we can derive the following constants of motion of (6) and

(33) from the sets (58) and (59):

$$I_1^{(0)}=\int_{-\infty}^{\infty}\tfrac{1}{2}u^{(0)2}\,\mathrm{d}x \qquad I_2^{(0)}=\int_{-\infty}^{\infty}[\tfrac{1}{3}(u^{(0)})^3-(u_1^{(0)})^2]\,\mathrm{d}x, \tag{60a, b}$$

$$I_3^{(0)}=\int_{-\infty}^{\infty}[2(u_2^{(0)})^2-\tfrac{10}{3}u^{(0)}(u_1^{(0)})^2+\tfrac{5}{18}(u^{(0)})^4]\,\mathrm{d}x, \qquad \text{etc}, \tag{60c}$$

$$I_1^{(1)}=\int_{-\infty}^{\infty}u^{(0)}u^{(1)}\,\mathrm{d}x, \tag{61a}$$

$$I_2^{(1)}=\int_{-\infty}^{\infty}[(u^{(0)})^2+2u_2^{(0)}]u^{(1)}\,\mathrm{d}x, \tag{61b}$$

$$I_3^{(1)}=\int_{-\infty}^{\infty}[4u_4^{(0)}+\tfrac{20}{3}u^{(0)}u_2^{(0)}+\tfrac{10}{3}(u_1^{(0)})^2+\tfrac{10}{9}(u^{(0)})^3]u^{(1)}\,\mathrm{d}x, \qquad \text{etc}, \tag{61c}$$

which are exactly the same as those obtained by Case and Roos (1981). We can also easily see that the constants in (60) and (61) are in involution such that

$$[I_l^{(0)}, I_{l'}^{(0)}]=0, \qquad l, l'=1, 2, 3, \ldots, \infty, \tag{62a}$$

$$[I_l^{(1)}, I_{l'}^{(1)}]=0, \qquad l, l'=1, 2, 3, \ldots, \infty, \tag{62b}$$

with respect to the Poisson bracket

$$[F, G]=\int_{-\infty}^{\infty}(\delta F/\delta u^{(1)}, \delta F/\delta u^{(0)})J\begin{pmatrix}\delta G/\delta u^{(1)}\\ \delta G/\delta u^{(0)}\end{pmatrix}\mathrm{d}x, \tag{63}$$

where J is as given in (54). Thus we have an infinite number of LB symmetries and commuting constants of motion for the first-order perturbation equation as well and therefore it is completely integrable.

5. The general case

Now it is also possible to find an infinite number of LB symmetries and constants of motion straightaway for all orders of perturbed evolution equations (3). The definitions (7)–(32) defined in § 2 will hold good here also.

We note that as a generalisation of the strong symmetry $\Phi^{(1)}$ in (47) we have an $(n+1)\times(n+1)$ matrix operator-valued function $\Phi^{(n)}$ such that it generates further symmetries from the known one satisfying the relation

$$\eta_{l+1}=\Phi^{(n)}\eta_l, \qquad l=1, 2, 3, \ldots, \infty, \tag{64}$$

where now

$$\eta_l(u)=(\eta_l^{(0)}(u), \eta_l^{(1)}(u), \ldots, \eta_l^{(n)}(u)),$$

$$\Phi^{(n)}=\begin{bmatrix}\Phi^{(0)}(u^{(0)}) & 0 & 0 & 0\\ \Phi^{(0)\prime}(u^{(1)}) & \Phi^{(0)}(u^{(0)}) & 0 & 0\\ \Phi^{(0)\prime}(u^{(2)}) & \Phi^{(0)\prime}(u^{(1)}) & \Phi^{(0)}(u^{(0)}) & 0\\ & & \vdots & \\ \Phi^{(0)\prime}(u^{(n)}) & \cdots & & \Phi^{(0)}(u^{(0)})\end{bmatrix} \tag{65}$$

As before the conserved covariants can be obtained from the relation

$$\gamma_{l+1} = \Phi^{(n)+}\gamma_l, \qquad l = 1, 2, 3, \ldots, \infty, \tag{66}$$

$$\gamma_l(u) = (\gamma_l^{(0)}(u), \gamma_l^{(1)}(u), \ldots, \gamma_l^{(n)}(u)), \tag{67}$$

and $\Phi^{(n)+}$ can be obtained from the relation

$$J\Phi^{(n)+} = \Phi^{(n)}J, \tag{68}$$

where the $(n+1)\times(n+1)$ matrix

$$J = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ 0 & D & 0 & \cdots & 0 \\ 0 & 0 & D & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & D \end{bmatrix}. \tag{69}$$

The corresponding constants of motion are given by the relation grad $\mathrm{I} = \gamma$ (see (23)). Thus we have shown that it is possible to derive an infinite number of LB symmetries and constants of motion for any order of perturbation as well. A detailed exposition and application of this theory to many other interesting soliton equations will be presented elsewhere.

References

Ablowitz M J and Segur H 1981 *Solitons and Inverse Scattering Transform* (Philadelphia: Siam)
Anderson R L and Ibragimov N Kh 1979 *Lie-Bäcklund Transformations in Applications* (Philadelphia: Siam)
Bullough R K and Caudrey P J 1980 *Solitons* (Heidelberg: Springer)
Case K M and Roos A M 1981 *J. Math. Phys.* **22** 2824
Fokas A S 1979 *Lett. Math. Phys.* **3** 467
—— 1980 *J. Math. Phys.* **21** 1318
Fuchssteiner B and Fokas A S 1981 *Physica* **4D** 47
Kumei S 1977 *J. Math. Phys.* **18** 256
Lakshmanan M 1978 *Phys. Lett.* **64A** 354
—— 1979 *J. Math. Phys.* **20** 1667
Strampp W 1982 *Lett. Math. Phys.* **6** 113
Tamizhmani K M and Lakshmanan M 1982 *Phys. Lett.* **90A** 159